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# Random walk on the Bethe lattice and hyperbolic Brownian motion

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**Abstract.** We give the exact solution to the problem of a random walk on the Bethe lattice through a mapping on an asymmetric random walk on the half-line. We also study the continuous limit of this model, and discuss in detail the relation between the random walk on the Bethe lattice and Brownian motion on a space of constant negative curvature.

## 1. Introduction

The Bethe lattice, or the infinite Cayley tree, presents a hierarchical structure that greatly simplifies some problems of statistical physics. It has therefore been widely used to obtain analytical results for problems that are otherwise intractable on Euclidean lattices. The physical relevance of these results is that the Bethe lattice is supposed to represent some mean field limit of Euclidean lattices of very large dimensions.

The tree structure is also directly relevant to characterize the phase space hierarchical structure of some disordered systems like spin glasses [1], where this structure plays an essential role in the understanding of the slow dynamics properties of these systems [2–4]

Random walks on Bethe lattices have already been studied either for their own interest [5–7], or in the context of polymer physics [8, 9], where counting unentangled loops of a polymer in an array of obstacles is equivalent to counting random walks on Cayley trees. In this paper, we give the exact solution to this problem of random walk on the Bethe lattice and discuss the correspondence with Brownian motion on a space of constant negative curvature.

The paper is organized as follows. In section 2, we consider a mapping between a random walk on the Bethe lattice and a biased random walk on the half-line with reflection at the origin. We derive asymptotic expressions at large time through a simple argument, and write the exact explicit solution using old results of Kac [10]. Section 3 is devoted to biased Brownian motion on the half-line with a reflective barrier at the origin, which represents the continuous limit of the previous discrete model. Finally in section 4, we discuss the relation between the Bethe lattice and hyperbolic geometry. We compare the properties of Brownian motion on a space of constant negative curvature to the results obtained in the previous sections.

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## 2. Random walk on the Bethe lattice and biased random walk on the half-chain

Hughes *et al* [5] have pointed out that a random walk on the Bethe lattice can be mapped onto a biased one-dimensional lattice walk with a reflecting barrier at the origin, provided one only considers the distance to the origin on the Bethe lattice as a function of time. In this section, we first re-derive this correspondence to fix our notations. We then use the generating function formalism and a Tauberian theorem to obtain the large time asymptotic behaviour of the probability of being at any given distance from the origin at time  $t$ . This result generalizes a previous one concerning the asymptotic behaviour of the probability of returning to the origin at time  $t$  [8, 6, 7]. We finally use an old result of Kac [10] to write the full solution of the problem of random walk on the Bethe lattice in an exact closed form.

### 2.1. Master equation

We consider a random walk on the Bethe lattice of coordination number  $z > 2$ . We take the origin of the random walk as the origin of coordinates, and call generation  $n$  the set of all the  $z(z-1)^{n-1}$  sites distant from the origin by  $n \geq 1$  bonds (see figure 1).

The random walk is defined as follows. At each time step, the particle jumps with probability  $(\frac{1}{z})$  to any of its  $z$  nearest neighbours. Therefore, the particle goes further from the origin with probability  $(\frac{z-1}{z})$ , and goes closer to the origin with probability  $(\frac{1}{z})$ . The corresponding *master equation* for the probability  $f_\tau(n)$  of being on any site of generation  $n$  after  $\tau$  time steps reads

$$f_{\tau+1}(n) = \left(1 - \frac{1}{z}\right) f_\tau(n-1) + \frac{1}{z} f_\tau(n+1) \quad \text{for } n \geq 2. \quad (1)$$

This has to be supplemented by specific rules at the boundary sites  $n = 1$  and  $n = 0$

$$f_{\tau+1}(1) = f_\tau(0) + \frac{1}{z} f_\tau(2) \quad (2)$$

$$f_{\tau+1}(0) = \frac{1}{z} f_\tau(1) \quad (3)$$

and by the initial condition at  $\tau = 0$

$$f_{\tau=0}(n) = \delta_{n,0}. \quad (4)$$

The normalization reads  $\sum_{n=0}^{\infty} f_\tau(n) = 1$  for any  $\tau$ .

The random walk on the Bethe lattice can therefore be described as an asymmetric one-dimensional random walk on the half-line  $n \geq 0$  with a reflecting barrier at the origin  $n = 0$  [5]. One may then use standard techniques for one-dimensional random walks.

### 2.2. Asymptotic behaviour at large time

It is convenient to introduce the generating function

$$F_n(\lambda) = \sum_{\tau=0}^{\infty} \lambda^\tau f_\tau(n). \quad (5)$$

The master equation (1) for  $f_\tau(n)$  is then converted into the recurrence relation

$$F_{n+1}(\lambda) = \frac{z}{\lambda} F_n(\lambda) - (z-1)F_{n-1}(\lambda) \quad \text{for } n \geq 2 \quad (6)$$

and the boundary rules (2), (3) together with the initial condition (4) give the relations

$$F_2(\lambda) = \frac{z}{\lambda} F_1(\lambda) - zF_0(\lambda) \quad (7)$$

$$F_1(\lambda) = \frac{z}{\lambda} [F_0(\lambda) - 1]. \quad (8)$$

Two linearly independent solutions of (6) are  $[r(\lambda)]^n$  and  $[r'(\lambda)]^n$  where  $r(\lambda)$  and  $r'(\lambda)$  are the solutions of the characteristic equation

$$r^2 - \frac{z}{\lambda}r + (z - 1) = 0. \tag{9}$$

As we must only keep the regular solution in the limit  $\lambda \rightarrow 0$  (5), we get for  $n \geq 2$

$$F_n(\lambda) = [r(\lambda)]^{n-1} F_1(\lambda) \quad \text{with} \quad r(\lambda) = \frac{z}{2\lambda} \left[ 1 - \sqrt{1 - \frac{4}{z^2}(z - 1)\lambda^2} \right]. \tag{10}$$

The two first terms  $F_1(\lambda)$  and  $F_0(\lambda)$  are then determined by (8)

$$F_0(\lambda) = \frac{2 \left( \frac{z-1}{z} \right)}{\left( \frac{z-2}{z} \right) + \sqrt{1 - \frac{4}{z^2}(z - 1)\lambda^2}} \tag{11}$$

$$F_1(\lambda) = \frac{z}{\lambda} \left( \frac{1 - \sqrt{1 - \frac{4}{z^2}(z - 1)\lambda^2}}{\left( \frac{z-2}{z} \right) + \sqrt{1 - \frac{4}{z^2}(z - 1)\lambda^2}} \right). \tag{12}$$

The probabilities  $f_\tau(n)$  may then in principle be obtained by expanding the previous expressions (10)–(12) in powers of  $\lambda$  (5). This has been done for the function  $F_0(\lambda)$  to obtain  $f_\tau(0)$  in an explicit form involving hypergeometric functions [8, 7]. This has not been done for the general case  $n > 0$ , where it is preferable to use some other method to get an exact explicit expression of  $f_\tau(n)$  for any  $n$  (see section 2.3). However, the asymptotic behaviour at large time of  $f_\tau(n)$  for any fixed  $n$  may be easily found from the generating function using a Tauberian theorem.

Introducing the notations  $\nu = \frac{z-2}{z}$  and  $\alpha = \frac{2}{z}\sqrt{z - 1}$ , the generating function reads (for  $n \geq 1$ )

$$F_n(\lambda) = 2 \left[ \frac{z}{2\lambda} \left( 1 - \sqrt{1 - (\alpha\lambda)^2} \right) \right]^n \frac{1}{\nu + \sqrt{1 - (\alpha\lambda)^2}} = g_n(\alpha\lambda) \tag{13}$$

with the auxiliary function

$$g_n(x) = 2 \left[ \frac{z\alpha}{2x} \left( 1 - \sqrt{1 - x^2} \right) \right]^n \frac{1}{\nu + \sqrt{1 - x^2}} \tag{14}$$

that admits the power series expansion

$$g_n(x) = \sum_{\{k \geq 0; (k-n) \text{ even}\}} c_k(n) x^k. \tag{15}$$

Since we have the simple relation  $f_\tau(n) = c_\tau(n)\alpha^\tau$ , the asymptotic behaviour of the probability  $f_\tau(n)$  at large time  $\tau$  is directly related to the behaviour of the coefficients  $c_k(n)$  for large order  $k$ . The latter can readily be obtained from the expansion of  $g_n$  (14) near  $x = 1$

$$g_n(e^{-E}) \underset{E \rightarrow 0^+}{\simeq} \frac{2}{\nu} \left( \frac{z\alpha}{2} \right)^n \left[ 1 - \left( n + \frac{1}{\nu} \right) \sqrt{2E} + O(E) \right]. \tag{16}$$

Using the power series expression (15), we get

$$\sum_{\{k \geq 0; (k-n) \text{ even}\}} c_k(n) (1 - e^{-kE}) \underset{E \rightarrow 0^+}{\simeq} \frac{2}{\nu} \left( n + \frac{1}{\nu} \right) \left( \frac{z\alpha}{2} \right)^n \sqrt{2E} + O(E). \tag{17}$$

The non-analyticity in  $\sqrt{E}$  implies that the coefficients  $c_k(n)$  present the following algebraic decay at large order  $k$  for  $(k - n)$  even:

$$c_k(n) \underset{k \rightarrow \infty}{\simeq} \frac{A(n)}{k^{3/2}}. \tag{18}$$

The prefactor  $A(n)$  can also be obtained from (17) by converting the sum into an integral:

$$A(n) = \frac{2}{v} \left(n + \frac{1}{v}\right) \left(\frac{z\alpha}{2}\right)^n \int_0^\infty du \frac{2^{3/2}}{\left(\frac{1-e^{-u}}{u^{3/2}}\right)} = \frac{2^{3/2}}{\sqrt{\pi}} \frac{(1+n\nu)}{\nu^2} \left(\frac{z\alpha}{2}\right)^n. \quad (19)$$

We finally get for  $(k-n)$  even

$$f_\tau(n) = c_\tau(n) \alpha^\tau \underset{\tau \rightarrow \infty}{\simeq} A(n) \frac{\alpha^\tau}{\tau^{3/2}} \quad (20)$$

or more explicitly

$$f_\tau(n) \underset{\tau \rightarrow \infty}{\simeq} \frac{2^{3/2}}{\sqrt{\pi}} \left(\frac{z}{z-2}\right)^2 \left(1 + n \frac{z-2}{z}\right) (\sqrt{z-1})^n \frac{1}{\tau^{3/2}} e^{-\tau \ln\left(\frac{z}{2\sqrt{z-1}}\right)}. \quad (21)$$

This formula, valid for  $n \geq 1$ , generalizes the result obtained previously for the case  $n = 0$  [8, 6, 7]

$$f_\tau(0) \underset{\tau \rightarrow \infty}{\simeq} \frac{2^{3/2}}{\sqrt{\pi}} \frac{z(z-1)}{(z-2)^2} \frac{1}{\tau^{3/2}} e^{-\tau \ln\left(\frac{z}{2\sqrt{z-1}}\right)}. \quad (22)$$

### 2.3. Exact expression for $f_\tau(n)$

In [10], Kac studied the asymmetric random walk (1) with the reflective boundary condition (2)–(3), but in the domain of negative drift  $1 < z < 2$ . Using our notations, the explicit expression obtained in this paper for the initial condition (4) reads

$$f_\tau^{(0 < z < 2)}(n) = (2-z) \left[ \delta_{n,0} + z(z-1)^{n-1} (1 - \delta_{n,0}) \right] \left[ \frac{1 + (-1)^{n+\tau}}{2} \right] \quad (23)$$

$$+ \frac{2}{\pi} \left[ \left(\frac{z-1}{z}\right) \delta_{n,0} + (z-1)^{n/2} (1 - \delta_{n,0}) \right] \left(\frac{2\sqrt{z-1}}{z}\right)^\tau I(n, \tau) \quad (24)$$

where

$$I(n, \tau) = [1 + (-1)^{n+\tau}] \int_0^{\frac{\pi}{2}} d\theta (\cos \theta)^\tau \frac{\tan^2 \theta}{\left(\frac{z-2}{z}\right)^2 + \tan^2 \theta} \times \left[ \cos n\theta + \left(\frac{z-2}{z}\right) \frac{\sin n\theta}{\sin \theta} \cos \theta \right]. \quad (25)$$

The first term (23) can be easily recovered as the normalized ‘stationary’ (invariant under the time translation  $\tau \rightarrow \tau + 2$ ) solution of (1) in the case of negative drift  $1 < z < 2$ .

For the case of positive drift  $z > 2$ , which we consider in this paper, there is no normalizable stationary solution. The exact expression for  $f_\tau$  describing the random walk on the Bethe lattice therefore only contains the second term (24)

$$f_\tau^{(z > 2)}(n) = \frac{2}{\pi} \left[ \left(\frac{z-1}{z}\right) \delta_{n,0} + (z-1)^{n/2} (1 - \delta_{n,0}) \right] \left(\frac{2\sqrt{z-1}}{z}\right)^\tau I(n, \tau). \quad (26)$$

For the particular case  $n = 0$ , the integral  $I(0, \tau)$  (25) may be computed through the change of variable  $x = \cos^2 \theta$

$$I(0, 2\tau) = 2 \int_0^{\frac{\pi}{2}} d\theta (\cos \theta)^{2\tau} \frac{\tan^2 \theta}{\left(\frac{z-2}{z}\right)^2 + \tan^2 \theta} = \int_0^1 dx \frac{x^{\tau-1} (1-x)^{1/2}}{1 - \left(\frac{4(z-1)}{z^2}\right)x} \quad (27)$$

which is a standard integral representation of the hypergeometric function

$$I(0, 2\tau) = B\left(\tau + \frac{1}{2}, \frac{3}{2}\right) F\left(1, \tau + \frac{1}{2}, \tau + 2, \frac{4(z-1)}{z^2}\right). \tag{28}$$

We therefore recover the expression given in [8, 7]

$$f_{\tau}^{(z>2)}(0) = \left(\frac{z-1}{z}\right) \left(\frac{\sqrt{z-1}}{z}\right)^{2\tau} \frac{\Gamma(2\tau+1)}{\Gamma(\tau+1)\Gamma(\tau+2)} \times F\left(1, \tau + \frac{1}{2}, \tau + 2, \frac{4(z-1)}{z^2}\right). \tag{29}$$

We may also recover from the expression (26) the asymptotic expression at large time given previously in (21). Indeed for large  $\tau$ , the integral  $I(n, \tau)$  (25) is dominated by the vicinity of  $\theta = 0$ . The estimation of the leading order

$$I(n, \tau) \underset{\tau \rightarrow \infty}{\simeq} [1 + (-1)^{n+\tau}] \int_0^\infty d\theta e^{-\tau \frac{\theta^2}{2}} \theta^2 \left[1 + \left(\frac{z-2}{z}\right)n\right] \tag{30}$$

$$= \left[\frac{1 + (-1)^{n+\tau}}{2}\right] \sqrt{2\pi} \left(\frac{z}{z-2}\right)^2 \left[1 + \left(\frac{z-2}{z}\right)n\right] \frac{1}{\tau^{\frac{3}{2}}} \tag{31}$$

gives back (21).

### 3. Biased Brownian motion on the half-line

We consider the continuum counterpart of the biased random walk described in section (1). The probability density  $P(x, t)$  for a Brownian particle of diffusion constant  $D$  submitted to a constant positive drift  $\mu$  satisfies the Fokker–Planck equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x} \left( D \frac{\partial P}{\partial x} - \mu P \right). \tag{32}$$

It must be supplemented by the reflection condition at  $x = 0$

$$\left( D \frac{\partial P}{\partial x} - \mu P \right) (x = 0, t) = 0 \tag{33}$$

to ensure conservation of probability, and by the initial condition

$$P(x, t \rightarrow 0^+) = \delta(x). \tag{34}$$

#### 3.1. Expression of the probability density $P(x, t)$

It is convenient to perform the transformation

$$P(x, t) = e^{\frac{\mu x}{2D}} \psi(x, t) \tag{35}$$

in order to cast the Fokker–Planck equation (32) into the Euclidean Schrödinger equation

$$\frac{\partial \psi}{\partial t} = -H\psi. \tag{36}$$

The corresponding Hamiltonian is a free one up to a constant shift:

$$H = -D \frac{d^2}{dx^2} + \frac{\mu^2}{4D}. \tag{37}$$

The boundary condition and the initial condition read respectively

$$\left(\frac{\partial \psi}{\partial x} - \frac{\mu}{2D} \psi\right)(x=0, t) = 0 \quad (38)$$

and

$$\psi(x, t \rightarrow 0^+) = \delta(x). \quad (39)$$

Using plane waves, it is easy to construct an orthonormal basis of eigenvectors  $\{\psi_k(x), k \in [0, +\infty[$

$$H \psi_k = \left(Dk^2 + \frac{\mu^2}{4D}\right) \psi_k \quad (40)$$

satisfying the boundary condition (38)

$$\psi_k(x) = \frac{1}{\sqrt{2\pi}} \left( e^{-ikx} - \frac{\mu + 2iDk}{\mu - 2iDk} e^{ikx} \right). \quad (41)$$

The Green function  $\psi(x, t)$  solution of (36)–(39) may be expanded onto this orthonormal basis

$$\psi(x, t) = \langle x | e^{-tH} | 0 \rangle = \int_0^{+\infty} dk \psi_k(x) \psi_k^*(0) e^{-t\left(Dk^2 + \frac{\mu^2}{4D}\right)}. \quad (42)$$

As a side remark, let us mention that in the case of negative drift  $\mu < 0$ , in addition to the continuous spectrum (41), there is also a bound state solution of zero energy which must be added to the expansion (42). This zero-energy state corresponds to the aforementioned stationary solution of Kac, the discrete case (23).

However, in the case of positive drift  $\mu > 0$  that we consider here, there is no stationary normalizable solution, and the spectrum is purely continuous (42). The resulting integral

$$\psi(x, t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dk \left( \frac{k^2 - ik\frac{\mu}{2D}}{k^2 + \frac{\mu^2}{4D^2}} \right) e^{ikx} e^{-t\left(Dk^2 + \frac{\mu^2}{4D}\right)} \quad (43)$$

can be computed through the following trick

$$\psi(x, t) = e^{-t\frac{\mu^2}{4D}} \left( -\frac{\partial^2}{\partial x^2} - \frac{\mu}{2D} \frac{\partial}{\partial x} \right) I(x). \quad (44)$$

The integral  $I(x)$ , being the Fourier transform of the product of a Gaussian by a Lorentzian, may be written as a convolution

$$\begin{aligned} I(x) &= \frac{1}{\pi} \int_{-\infty}^{+\infty} dk e^{ikx} \left( \frac{1}{k^2 + \frac{\mu^2}{4D^2}} \right) e^{-tDk^2} \\ &= \frac{D}{\mu} \int_{-\infty}^{+\infty} dy e^{-\frac{\mu}{2D}|x-y|} \frac{1}{\sqrt{\pi Dt}} e^{-\frac{y^2}{4Dt}}. \end{aligned} \quad (45)$$

Going back to the probability density (35), we finally obtain

$$P(x, t) = \frac{e^{-t\frac{\mu^2}{4D}}}{\sqrt{\pi Dt}} e^{x\frac{\mu}{2D}} \left[ e^{-\frac{x^2}{4Dt}} - \frac{\mu}{2D} \int_x^\infty dy e^{-\frac{\mu}{2D}(y-x)} e^{-\frac{y^2}{4Dt}} \right]. \quad (46)$$

It is useful to transform the previous expression into

$$P(x, t) = \frac{e^{-t\frac{\mu^2}{4D}}}{2\sqrt{\pi}(Dt)^{3/2}} e^{\frac{\mu}{2D}x} \int_x^\infty dy y e^{-\frac{\mu}{2D}(y-x)} e^{-\frac{y^2}{4Dt}}. \quad (47)$$

The asymptotic behaviour at large time  $t$  immediately follows

$$P(x, t) \underset{t \rightarrow \infty}{\simeq} \frac{e^{-t \frac{\mu^2}{4D}}}{2\sqrt{\pi}(Dt)^{3/2}} e^{\frac{\mu}{D}x} \int_x^\infty dy y e^{-\frac{\mu}{2D}y} \tag{48}$$

$$= 2\sqrt{\frac{D}{\pi}} \frac{1}{\mu^2} \left(1 + x \frac{\mu}{2D}\right) e^{\frac{\mu}{2D}x} \frac{1}{t^{3/2}} e^{-t \frac{\mu^2}{4D}}. \tag{49}$$

Comparison with the corresponding result for the discrete case (21)

$$f_\tau(n) \underset{\tau \rightarrow \infty}{\simeq} \frac{2^{3/2}}{\sqrt{\pi}} \left(\frac{z}{z-2}\right)^2 \left(1 + n \frac{z-2}{z}\right) e^{n \ln \sqrt{z-1}} \frac{1}{\tau^{3/2}} e^{-\tau \ln\left(\frac{z}{2\sqrt{z-1}}\right)} \tag{50}$$

shows that even if the two expressions behave qualitatively the same as  $t^{-3/2}e^{-At}$  as a function of time, and as  $(1+nC)e^{nB}$  as a function of space, there is no direct detailed correspondence at this stage. If one naively tries to identify  $x = n$  and  $t = \tau$ , then there is no way to find two functions  $\mu(z)$  and  $D(z)$  to make equations (49) and (50) identical. The relation between the random walk on the Bethe lattice and the continuous model (32) studied in this section therefore needs further clarification.

### 3.2. Discussion of the continuous limit

To define properly the continuous limit of the random walk on the Bethe lattice, we must introduce some unit length  $\Delta x$  and some unit time  $\Delta t$  into the master equation (1)

$$f(x, t + \Delta t) = \left(1 - \frac{1}{z}\right) f(x - \Delta x, t) + \frac{1}{z} f(x + \Delta x, t). \tag{51}$$

The Taylor expansion

$$\frac{\partial f}{\partial t} = \left(\frac{(\Delta x)^2}{2\Delta t}\right) \frac{\partial^2 f}{\partial x^2} - \left(\frac{z-2}{z} \frac{\Delta x}{\Delta t}\right) \frac{\partial f}{\partial x} \tag{52}$$

gives the Fokker–Planck equation (32) in the limit

$$\Delta x \rightarrow 0^+ \quad \Delta t \rightarrow 0^+ \quad z \rightarrow 2^+ \tag{53}$$

with

$$\frac{(\Delta x)^2}{2\Delta t} = D \quad \text{and} \quad \frac{z-2}{z} \frac{\Delta x}{\Delta t} = \mu \tag{54}$$

where  $D$  and  $\mu$  are fixed. This continuous limit therefore requires an analytic continuation to non-integer coordination number  $z$  in order to take the limit  $z \rightarrow 2^+$ . With the prescription (53)–(54), the result (21) for the discrete case indeed corresponds to the result (49) of the continuous case.

Let us now write for the discrete case the analogue of the transformations that we performed for the continuous case in order to establish some correspondence with the work of Clark *et al* [11]. The analogue of the transformation (35) to make the walk (1) symmetric

$$f_\tau(n) = e^{n \ln \sqrt{z-1}} \tilde{f}_\tau(n) \tag{55}$$

and the extraction of the shift factor analogue to the shift present in (37)

$$\tilde{f}_\tau(n) = e^{-\tau \ln\left(\frac{z}{2\sqrt{z-1}}\right)} p_\tau(n) \tag{56}$$

transform the drifted random walk with reflection at the origin (1)–(3) into a symmetric random walk with a partial absorption at the origin

$$p_{\tau+1}(n) = \frac{1}{2} p_{\tau}(n-1) + \frac{1}{2} p_{\tau}(n+1) \quad \text{for } n \geq 2 \quad (57)$$

$$p_{\tau+1}(1) = \gamma p_{\tau}(0) + \frac{1}{2} p_{\tau}(2) \quad (58)$$

$$p_{\tau+1}(0) = \frac{1}{2} p_{\tau}(1) \quad (59)$$

where  $\gamma = \frac{z}{2(z-1)}$  denotes the reflection coefficient at the origin. The continuous limit of this random walk with partial absorption at the origin is indeed a quantum mechanical free problem on the half-line with some mixed boundary condition (38). The detailed discussion of this point contained in [11] is equivalent to our approach (see 53–54).

The meaning of the continuous model (32) to represent the random walk on the Bethe lattice is now clear. We will now present another continuous model where the effective drift comes from the geometry of the underlying space itself.

#### 4. Bethe lattices and hyperbolic geometry

A link between Cayley trees and hyperbolic geometry has already been emphasized in [12, 13] through the introduction of tessalations. Indeed, for any integer  $z \geq 3$ , one can construct a tessalation of the Poincaré upper half-plane with polygons with  $z$  sides. Joining the centres of adjacent polygons gives a Bethe lattice of coordination number  $z$ . In the following, we will not use this approach, but rather establish some intrinsic link between Bethe lattices and spaces of constant negative curvature without introducing any tessalation.

##### 4.1. Hyperbolic geometry

An  $N$ -dimensional Riemannian manifold of constant negative Gaussian curvature  $K = -\frac{1}{a^2}$  may be described by the metric

$$ds_N^2 = dr^2 + a^2 \sinh^2 \frac{r}{a} d\sigma_{N-1}^2 \quad (60)$$

where  $r \in [0, +\infty]$  measures the distance to the origin, and where  $d\sigma_{N-1}^2$  denotes the metric of the unit-sphere  $S_{N-1}$ . For example,  $d\sigma_1^2 = d\theta^2$  is the metric of the unit circle in terms of polar angle  $\theta$ , and  $d\sigma_2^2 = d\theta^2 + \sin^2 \theta d\phi^2$  is the metric of the unit sphere  $S_2$  in terms of the spherical angles  $(\theta, \phi)$ . The volume element  $dV$  is covariantly defined as

$$dV_N = \left( a \sinh \frac{r}{a} \right)^{N-1} dr d\Omega_{N-1} \quad (61)$$

where  $d\Omega_{N-1}$  is the surface element of the unit-sphere  $S_{N-1}$ ; for example,  $d\Omega_1 = d\theta$  and  $d\Omega_2 = \sin \theta d\theta d\phi$ .

In particular, the volume of a ball of radius  $R$  reads

$$V_N(R) = \Omega_{N-1} a^N \int_0^{\frac{R}{a}} dx (\sinh x)^{N-1} \quad (62)$$

where  $\Omega_{N-1}$  is the total surface of the unit-sphere  $S_{N-1}$ ; for example,  $\Omega_1 = 2\pi$  and  $\Omega_2 = 4\pi$ . This volume has the important property to grow exponentially with the distance  $R$

$$V_N(R) \underset{R \gg a}{\propto} e^{R \left( \frac{N-1}{a} \right)} \quad (63)$$

in contrast with the power dependance  $R^N$  in Euclidean space of dimension  $N$ .

In the same way, the number of sites on the Bethe lattice up to generation  $n$

$$\begin{aligned} \mathcal{N}(n) &= 1 + z + z(z-1) + z(z-1)^2 + \dots + z(z-1)^{n-1} \\ &= \frac{z(z-1)^n - 2}{z-2} \end{aligned} \tag{64}$$

grows exponentially with the number  $n$  of generations

$$\mathcal{N}(n) \underset{n \gg 1}{\propto} e^{n \ln(z-1)}. \tag{65}$$

This is why the Bethe lattice is often considered to be like a Euclidean lattice of infinite dimension. But it is certainly more interesting to relate it to hyperbolic geometry which presents the same property (63). More precisely, we may introduce some lattice spacing  $\Delta r$  on the Cayley tree so that the sites of generation  $n$  are at distance  $r = n\Delta r$  from the origin. Then the exponential dependence of (65) corresponds to (63) in the following continuous limit of the Bethe lattice

$$\Delta r \rightarrow 0^+ \qquad n \rightarrow \infty \qquad z \rightarrow 2^+ \tag{66}$$

with

$$n\Delta r = r \qquad \text{and} \qquad \frac{z-2}{\Delta r} = \frac{N-1}{a} \tag{67}$$

where  $N$  and  $a$  are fixed.

#### 4.2. Hyperbolic Brownian motion

The radial part  $\Delta_r$  of the Laplace operator on the  $N$ -dimensional Riemannian manifold of constant negative Gaussian curvature defined by the metric (60) reads

$$\Delta_r = \frac{1}{\left(\sinh \frac{r}{a}\right)^{N-1}} \frac{\partial}{\partial r} \left[ \left(\sinh \frac{r}{a}\right)^{N-1} \frac{\partial}{\partial r} \right]. \tag{68}$$

On this manifold, free Brownian motion starting from the origin is defined by the diffusion equation for the Green's function  $G_N(r, t)$

$$\frac{\partial G_N}{\partial t} = D \Delta_r G_N \tag{69}$$

and the initial condition

$$G_N(r, t) \underset{t \rightarrow 0^+}{\longrightarrow} \delta(r) \frac{1}{r \Omega_{N-1}}. \tag{70}$$

The normalization of the Green function  $G_N(r, t)$  then reads for any time  $t$

$$1 = \int dS G_N(r, t) = \Omega_{N-1} \int_0^{+\infty} dr \left(a \sinh \frac{r}{a}\right)^{N-1} G_N(r, t). \tag{71}$$

The solution reads respectively in two and three dimensions

$$G_2(r, t) = \frac{e^{-\frac{Dt}{4a^2}}}{4\sqrt{2}a(\pi Dt)^{3/2}} \int_r^\infty dy \frac{y e^{-\frac{y^2}{4Dt}}}{\sqrt{\cosh\left(\frac{y}{a}\right) - \cosh\left(\frac{r}{a}\right)}} \tag{72}$$

and

$$G_3(r, t) = \frac{e^{-\frac{Dt}{a^2}}}{8a(\pi Dt)^{3/2}} \frac{r e^{-\frac{r^2}{4Dt}}}{\sinh \frac{r}{a}}. \tag{73}$$

Consider now the probability density  $P_N(r, t)$  to be at time  $t$  at a distance  $r$  from the origin

$$P_t(r) = S_{N-1} \left( a \sinh \frac{r}{a} \right)^{N-1} G_N(r, t) \quad (74)$$

normalized with respect to the flat measure  $dr$  (71)

$$1 = \int_0^{+\infty} dr P_N(r, t). \quad (75)$$

This probability density satisfies the Fokker–Planck equation (69)

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial r} \left[ D \frac{\partial P}{\partial r} - D \left( \frac{N-1}{a} \right) \coth \left( \frac{r}{a} \right) P \right] \quad (76)$$

and the initial condition at  $t = 0$  (70)

$$P_N(r, t) \xrightarrow[t \rightarrow 0^+]{} \delta(r). \quad (77)$$

The solution reads respectively in two and three dimensions (72)–(73)

$$P_2(r, t) = \frac{e^{-\frac{Dt}{4a^2}}}{2\sqrt{2\pi}(Dt)^{3/2}} \sinh \left( \frac{r}{a} \right) \int_r^\infty dy \frac{y e^{-\frac{y^2}{4Dt}}}{\sqrt{\cosh \left( \frac{y}{a} \right) - \cosh \left( \frac{r}{a} \right)}} \quad (78)$$

$$P_3(r, t) = a \frac{e^{-\frac{Dt}{a^2}}}{2\sqrt{\pi}(Dt)^{3/2}} r \sinh \frac{r}{a} e^{-\frac{r^2}{4Dt}}. \quad (79)$$

At large distance from the origin  $r \gg a$ , the Fokker–Planck equation (76) corresponds to a one-dimensional diffusion with a constant drift  $\mu = D \frac{N-1}{a}$  (32)

$$\frac{\partial P}{\partial t} \sim \frac{\partial}{\partial r} \left[ D \frac{\partial P}{\partial r} - D \left( \frac{N-1}{a} \right) P \right]. \quad (80)$$

One may check that this identification of the effective radial constant drift  $\mu = D \frac{N-1}{a}$  at large distance on the hyperbolic space is entirely consistent with the two continuous limits of the Bethe lattice previously defined in (53)–(54) and (66)–(67).

The solutions (78)–(79) in two and three dimensions read approximatively at large distance

$$P_2(r, t) \underset{r \gg a}{\simeq} \frac{e^{-\frac{Dt}{4a^2}}}{4\sqrt{\pi}(Dt)^{3/2}} e^{\frac{r}{2a}} \int_r^\infty dy y e^{-\frac{y-r}{2a}} e^{-\frac{y^2}{4Dt}} \quad (81)$$

$$P_3(r, t) \underset{r \gg a}{\simeq} a \frac{e^{-\frac{Dt}{a^2}}}{4\sqrt{\pi}(Dt)^{3/2}} r e^{\frac{r}{a}} e^{-\frac{r^2}{4Dt}} \quad (82)$$

to be compared with (47). These expressions at large time

$$P_2(r \gg a, t \rightarrow \infty) \simeq \frac{a}{2\sqrt{\pi}D^{3/2}} r e^{\frac{r}{2a}} t^{-3/2} e^{-\frac{Dt}{4a^2}} \quad (83)$$

$$P_3(r, t) \underset{r \gg a}{\simeq} \frac{a}{4\sqrt{\pi}D^{3/2}} r e^{\frac{r}{a}} t^{-3/2} e^{-\frac{Dt}{a^2}} \quad (84)$$

are the exact analogue of expression (49) at large distance

$$P(x \gg \frac{2D}{\mu}, t \rightarrow \infty) = \frac{1}{2\mu\sqrt{\pi}D} x e^{\frac{\mu}{2D}x} t^{-3/2} e^{-t\frac{\mu^2}{4D}} \quad (85)$$

for  $\mu = D \frac{N-1}{a}$ . This confirms the equivalence with biased Brownian motion on the half-line.

## 5. Conclusion

The exact solution (26) to the homogenous random walk on the Bethe lattice was obtained through a mapping onto a biased one-dimensional random walk on the half-line that describes the ‘radial’ dynamics on the tree. This mapping to a one-dimensional lattice can of course be extended to inhomogenous random walks on the Bethe lattice that still preserve the radial invariance on the tree [5, 14, 15], but does not hold any longer as soon as two sites belonging to the same generation are no longer equivalent. This is in particular the case when one considers disordered problems on the Bethe lattice. This is why the relation with hyperbolic geometry that we presented in section 4 is much more profound. On a Riemannian manifold of constant negative curvature, we have at our disposal not only the radial coordinate that corresponds to the generation number on the Bethe lattice, but also angular coordinates, that correspond to the ‘degree of freedom’ inside a given generation of the tree. In this paper we have discussed in detail the radial correspondence, but it would certainly be interesting to also consider more precisely the angular part, and to study in particular the continuous limit of some disordered systems on the Bethe lattice.

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## References

- [1] Mézard M, Parisi G and Virasoro M A 1986 *Spin Glass Theory and Beyond* (Singapore: World Scientific)
- [2] Paladin G, Mézard M and de Dominicis C 1985 *J. Physique Lett.* **46** L985
- [3] Sibani P and Hoffmann K H 1987 *Europhys. Lett.* **4** 967
- [4] Bouchaud J P and Dean D S 1995 *J. Physique I* **5** 265
- [5] Hughes B D and Sahimi M 1982 *J. Stat. Phys.* **29** 781
- [6] Cassi D 1989 *Europhys. Lett.* **9** 627
- [7] Giacometti A 1995 *J. Phys. A: Math. Gen.* **28** L13
- [8] Helfand E and Pearson D S 1983 *J. Chem. Phys.* **79** 2054
- [9] Nechaev S K, Semenov A N and Koleva M K 1987 *Physica* **140A** 506
- [10] Kac M 1947 *Am. Math. Monthly* **54** 369 (with misprints) reprinted in 1954 *Selected Papers on Noise and Stochastic Processes* ed N Wax (New York: Dover)
- [11] Clark T E, Menikoff R and Sharp D H 1980 *Phys. Rev. D* **22** 3012
- [12] Mosseri R and Sadoc J F 1982 *J. Physique Lett.* **43** L249
- [13] Khokhlov A R and Nechaev S K 1985 *Phys. Lett.* **112A** 156
- [14] Cassi D 1990 *Europhys. Lett.* **13** 583
- [15] Aslangul C, Barthélémy M, Pottier N and Saint-James D 1991 *Europhys. Lett.* **15** 251